

# CONTINUITY AND DIFFERENTIABILITY

## 5.1 Overview

### 5.1.1 Continuity of a function at a point

Let  $f$  be a real function on a subset of the real numbers and let  $c$  be a point in the domain of  $f$ . Then  $f$  is continuous at  $c$  if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at  $x = c$  exist and are equal to each other, i.e.,

$$\lim_{x \rightarrow c^-} f(x) = f(c) = \lim_{x \rightarrow c^+} f(x)$$

then  $f$  is said to be continuous at  $x = c$ .

### 5.1.2 Continuity in an interval

(i)  $f$  is said to be continuous in an open interval  $(a, b)$  if it is continuous at every point in this interval.

(ii)  $f$  is said to be continuous in the closed interval  $[a, b]$  if

- $f$  is continuous in  $(a, b)$

- $\lim_{x \rightarrow a^+} f(x) = f(a)$

- $\lim_{x \rightarrow b^-} f(x) = f(b)$

### 5.1.3 Geometrical meaning of continuity

- (i) Function  $f$  will be continuous at  $x = c$  if there is no break in the graph of the function at the point  $(c, f(c))$ .
- (ii) In an interval, function is said to be continuous if there is no break in the graph of the function in the entire interval.

### 5.1.4 Discontinuity

The function  $f$  will be discontinuous at  $x = a$  in any of the following cases :

- (i)  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  exist but are not equal.
- (ii)  $\lim_{x \rightarrow a^-} f(x)$  and  $\lim_{x \rightarrow a^+} f(x)$  exist and are equal but not equal to  $f(a)$ .
- (iii)  $f(a)$  is not defined.

### 5.1.5 Continuity of some of the common functions

Function $f(x)$	Interval in which $f$ is continuous
1. The constant function, i.e. $f(x) = c$	<b>R</b>
2. The identity function, i.e. $f(x) = x$	
3. The polynomial function, i.e. $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$	
4. $ x - a $	$(-\infty, \infty)$
5. $x^n$ , $n$ is a positive integer	$(-\infty, \infty) - \{0\}$
6. $p(x) / q(x)$ , where $p(x)$ and $q(x)$ are polynomials in $x$	<b>R</b> - $\{x : q(x) = 0\}$
7. $\sin x, \cos x$	<b>R</b>
8. $\tan x, \sec x$	<b>R</b> - $\{(2n + 1) \frac{\pi}{2} : n \in \mathbf{Z}\}$
9. $\cot x, \operatorname{cosec} x$	<b>R</b> - $\{n\pi : n \in \mathbf{Z}\}$

- |  |                                |
|--|--------------------------------|
| 10. $e^x$  | <b>R</b>                       |
| 11. $\log x$   | $(0, \infty)$                  |
| 12. The inverse trigonometric functions,<br>i.e., $\sin^{-1} x$ , $\cos^{-1} x$ etc. | In their respective<br>domains |

### 5.1.6 Continuity of composite functions

Let  $f$  and  $g$  be real valued functions such that  $(f \circ g)$  is defined at  $a$ . If  $g$  is continuous at  $a$  and  $f$  is continuous at  $g(a)$ , then  $(f \circ g)$  is continuous at  $a$ .

### 5.1.7 Differentiability

The function defined by  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ , wherever the limit exists, is defined to be the derivative of  $f$  at  $x$ . In other words, we say that a function  $f$  is differentiable at a point  $c$  in its domain if both  $\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h}$ , called left hand derivative, denoted by  $Lf'(c)$ , and  $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$ , called right hand derivative, denoted by  $Rf'(c)$ , are finite and equal.

- (i) The function  $y = f(x)$  is said to be differentiable in an open interval  $(a, b)$  if it is differentiable at every point of  $(a, b)$
- (ii) The function  $y = f(x)$  is said to be differentiable in the closed interval  $[a, b]$  if  $Rf'(a)$  and  $Lf'(b)$  exist and  $f'(x)$  exists for every point of  $(a, b)$ .
- (iii) Every differentiable function is continuous, but the converse is not true

### 5.1.8 Algebra of derivatives

If  $u, v$  are functions of  $x$ , then

$$(i) \quad \frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \qquad (ii) \quad \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$(iii) \quad \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

**5.1.9** Chain rule is a rule to differentiate composition of functions. Let  $f = v \circ u$ . If

$$t = u(x) \text{ and both } \frac{dt}{dx} \text{ and } \frac{dv}{dt} \text{ exist then } \frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

**5.1.10** Following are some of the standard derivatives (in appropriate domains)

$$1. \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad 2. \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$3. \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \quad 4. \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$5. \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$$

$$6. \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$$

### 5.1.11 Exponential and logarithmic functions

- (i) The exponential function with positive base  $b > 1$  is the function  $y = f(x) = b^x$ . Its domain is  $\mathbf{R}$ , the set of all real numbers and range is the set of all positive real numbers. Exponential function with base 10 is called the common exponential function and with base  $e$  is called the natural exponential function.
- (ii) Let  $b > 1$  be a real number. Then we say logarithm of  $a$  to base  $b$  is  $x$  if  $b^x = a$ . Logarithm of  $a$  to the base  $b$  is denoted by  $\log_b a$ . If the base  $b = 10$ , we say it is common logarithm and if  $b = e$ , then we say it is natural logarithms.  $\log x$  denotes the logarithm function to base  $e$ . The domain of logarithm function is  $\mathbf{R}^+$ , the set of all positive real numbers and the range is the set of all real numbers.
- (iii) The properties of logarithmic function to any base  $b > 1$  are listed below:

$$1. \log_b(xy) = \log_b x + \log_b y$$

$$2. \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y$$

$$3. \log_b x^n = n \log_b x$$

$$4. \log_b x = \frac{\log_c x}{\log_c b}, \text{ where } c > 1$$

$$5. \log_b x = \frac{1}{\log_x b}$$

$$6. \log_b b = 1 \text{ and } \log_b 1 = 0$$

(iv) The derivative of  $e^x$  w.r.t.,  $x$  is  $e^x$ , i.e.  $\frac{d}{dx}(e^x) = e^x$ . The derivative of  $\log x$

$$\text{w.r.t., } x \text{ is } \frac{1}{x}; \text{ i.e. } \frac{d}{dx}(\log x) = \frac{1}{x}.$$

**5.1.12** Logarithmic differentiation is a powerful technique to differentiate functions of the form  $f(x) = (u(x))^{v(x)}$ , where both  $f$  and  $u$  need to be positive functions for this technique to make sense.

**5.1.13** Differentiation of a function with respect to another function

Let  $u = f(x)$  and  $v = g(x)$  be two functions of  $x$ , then to find derivative of  $f(x)$  w.r.t.

to  $g(x)$ , i.e., to find  $\frac{du}{dv}$ , we use the formula

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

**5.1.14** *Second order derivative*

$\frac{d}{dx} \frac{dy}{dx} = \frac{d^2y}{dx^2}$  is called the second order derivative of  $y$  w.r.t.  $x$ . It is denoted by  $y''$  or  $y_2$ , if  $y = f(x)$ .

**5.1.15** *Rolle's Theorem*

Let  $f: [a, b] \rightarrow \mathbf{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ , such that  $f(a) = f(b)$ , where  $a$  and  $b$  are some real numbers. Then there exists at least one point  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .

Geometrically Rolle's theorem ensures that there is at least one point on the curve  $y = f(x)$  at which tangent is parallel to  $x$ -axis (abscissa of the point lying in  $(a, b)$ ).

### 5.1.16 Mean Value Theorem (Lagrange)

Let  $f: [a, b] \rightarrow \mathbf{R}$  be a continuous function on  $[a, b]$  and differentiable on  $(a, b)$ . Then

there exists at least one point  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .

Geometrically, Mean Value Theorem states that there exists at least one point  $c$  in  $(a, b)$  such that the tangent at the point  $(c, f(c))$  is parallel to the secant joining the points  $(a, f(a))$  and  $(b, f(b))$ .

## 5.2 Solved Examples

### Short Answer (S.A.)

**Example 1** Find the value of the constant  $k$  so that the function  $f$  defined below is

continuous at  $x = 0$ , where  $f(x) = \begin{cases} \frac{1 - \cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$ .

**Solution** It is given that the function  $f$  is continuous at  $x = 0$ . Therefore,  $\lim_{x \rightarrow 0} f(x) = f(0)$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{8x^2} = k$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{2 \sin^2 2x}{8x^2} = k$$

$$\Rightarrow \lim_{x \rightarrow 0} \left( \frac{\sin 2x}{2x} \right)^2 = k$$

$$\Rightarrow k = 1$$

Thus,  $f$  is continuous at  $x = 0$  if  $k = 1$ .

**Example 2** Discuss the continuity of the function  $f(x) = \sin x \cdot \cos x$ .

**Solution** Since  $\sin x$  and  $\cos x$  are continuous functions and product of two continuous function is a continuous function, therefore  $f(x) = \sin x \cdot \cos x$  is a continuous function.

**Example 3** If  $f(x) = \begin{cases} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & x \neq 2 \\ k & , x = 2 \end{cases}$  is continuous at  $x = 2$ , find

the value of  $k$ .

**Solution** Given  $f(2) = k$ .

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2} \\ &= \lim_{x \rightarrow 2} \frac{(x+5)(x-2)^2}{(x-2)^2} = \lim_{x \rightarrow 2} (x+5) = 7 \end{aligned}$$

As  $f$  is continuous at  $x = 2$ , we have

$$\begin{aligned} \lim_{x \rightarrow 2} f(x) &= f(2) \\ \Rightarrow k &= 7. \end{aligned}$$

**Example 4** Show that the function  $f$  defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at  $x = 0$ .

**Solution** Left hand limit at  $x = 0$  is given by

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x \sin \frac{1}{x} = 0 \quad \left[ \text{since, } -1 < \sin \frac{1}{x} < 1 \right]$$

Similarly  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x \sin \frac{1}{x} = 0$ . Moreover  $f(0) = 0$ .

Thus  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$ . Hence  $f$  is continuous at  $x = 0$

**Example 5** Given  $f(x) = \frac{1}{x-1}$ . Find the points of discontinuity of the composite function  $y = f[f(x)]$ .

**Solution** We know that  $f(x) = \frac{1}{x-1}$  is discontinuous at  $x = 1$

Now, for  $x \neq 1$ ,

$$f(f(x)) = f\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1}-1} = \frac{x-1}{2-x},$$

which is discontinuous at  $x = 2$ .

Hence, the points of discontinuity are  $x = 1$  and  $x = 2$ .

**Example 6** Let  $f(x) = x|x|$ , for all  $x \in \mathbf{R}$ . Discuss the derivability of  $f(x)$  at  $x = 0$

**Solution** We may rewrite  $f$  as  $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

$$\text{Now } Lf'(0) = \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{-h^2 - 0}{h} = \lim_{h \rightarrow 0^-} -h = 0$$

$$\text{Now } Rf'(0) = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^2 - 0}{h} = \lim_{h \rightarrow 0^+} h = 0$$

Since the left hand derivative and right hand derivative both are equal, hence  $f$  is differentiable at  $x = 0$ .

**Example 7** Differentiate  $\sqrt{\tan \sqrt{x}}$  w.r.t.  $x$

**Solution** Let  $y = \sqrt{\tan \sqrt{x}}$ . Using chain rule, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2\sqrt{\tan \sqrt{x}}} \cdot \frac{d}{dx}(\tan \sqrt{x}) \\ &= \frac{1}{2\sqrt{\tan \sqrt{x}}} \cdot \sec^2 \sqrt{x} \frac{d}{dx}(\sqrt{x}) \\ &= \frac{1}{2\sqrt{\tan \sqrt{x}}} (\sec^2 \sqrt{x}) \frac{1}{2\sqrt{x}} \\ &= \frac{(\sec^2 \sqrt{x})}{4\sqrt{x}\sqrt{\tan \sqrt{x}}}. \end{aligned}$$

**Example 8** If  $y = \tan(x + y)$ , find  $\frac{dy}{dx}$ .

**Solution** Given  $y = \tan(x + y)$ . Differentiating both sides w.r.t.  $x$ , we have



$$\begin{aligned}\frac{dy}{dx} &= \sec^2(x+y) \frac{d}{dx}(x+y) \\ &= \sec^2(x+y) \left(1 + \frac{dy}{dx}\right)\end{aligned}$$

$$\text{or } [1 - \sec^2(x+y)] \frac{dy}{dx} = \sec^2(x+y)$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{\sec^2(x+y)}{1 - \sec^2(x+y)} = -\operatorname{cosec}^2(x+y).$$

**Example 9** If  $e^x + e^y = e^{x+y}$ , prove that

$$\frac{dy}{dx} = -e^{y-x}.$$

**Solution** Given that  $e^x + e^y = e^{x+y}$ . Differentiating both sides w.r.t.  $x$ , we have

$$e^x + e^y \frac{dy}{dx} = e^{x+y} \left(1 + \frac{dy}{dx}\right)$$

$$\text{or } (e^y - e^{x+y}) \frac{dy}{dx} = e^{x+y} - e^x,$$

$$\text{which implies that } \frac{dy}{dx} = \frac{e^{x+y} - e^x}{e^y - e^{x+y}} = \frac{e^x + e^y - e^x}{e^y - e^x - e^y} = -e^{y-x}.$$

**Example 10** Find  $\frac{dy}{dx}$ , if  $y = \tan^{-1} \left( \frac{3x - x^3}{1 - 3x^2} \right)$ ,  $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$ .

**Solution** Put  $x = \tan \theta$ , where  $-\frac{\pi}{6} < \theta < \frac{\pi}{6}$ .

$$\begin{aligned}\text{Therefore, } y &= \tan^{-1} \left( \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right) \\ &= \tan^{-1} (\tan 3\theta) \\ &= 3\theta \quad \left( \text{because } -\frac{\pi}{2} < 3\theta < \frac{\pi}{2} \right) \\ &= 3 \tan^{-1} x\end{aligned}$$

Hence, 
$$\frac{dy}{dx} = \frac{3}{1+x^2}.$$

**Example 11** If  $y = \sin^{-1}\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\}$  and  $0 < x < 1$ , then find  $\frac{dy}{dx}$ .

**Solution** We have  $y = \sin^{-1}\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\}$ , where  $0 < x < 1$ .

Put  $x = \sin A$  and  $\sqrt{x} = \sin B$

Therefore, 
$$\begin{aligned} y &= \sin^{-1}\{\sin A\sqrt{1-\sin^2 B} - \sin B\sqrt{1-\sin^2 A}\} \\ &= \sin^{-1}\{\sin A \cos B - \sin B \cos A\} \\ &= \sin^{-1}\{\sin(A-B)\} = A - B \end{aligned}$$

Thus  $y = \sin^{-1} x - \sin^{-1} \sqrt{x}$

Differentiating w.r.t.  $x$ , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-\sqrt{x}}^2} \cdot \frac{d(\sqrt{x})}{dx} \\ &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\sqrt{1-x}}. \end{aligned}$$

**Example 12** If  $x = a \sec^3 \theta$  and  $y = a \tan^3 \theta$ , find  $\frac{dy}{dx}$  at  $\theta = \frac{\pi}{3}$ .

**Solution** We have  $x = a \sec^3 \theta$  and  $y = a \tan^3 \theta$ .

Differentiating w.r.t.  $\theta$ , we get

$$\begin{aligned} \frac{dx}{d\theta} &= 3a \sec^2 \theta \frac{d}{d\theta}(\sec \theta) = 3a \sec^3 \theta \tan \theta \\ \text{and } \frac{dy}{d\theta} &= 3a \tan^2 \theta \frac{d}{d\theta}(\tan \theta) = 3a \tan^2 \theta \sec^2 \theta. \end{aligned}$$

Thus 
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \tan^2 \theta \sec^2 \theta}{3a \sec^3 \theta \tan \theta} = \frac{\tan \theta}{\sec \theta} = \sin \theta.$$

Hence,  $\frac{dy}{dx}$  at  $\theta = \frac{\pi}{3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ .

**Example 13** If  $x^y = e^{x-y}$ , prove that  $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$ .

**Solution** We have  $x^y = e^{x-y}$ . Taking logarithm on both sides, we get

$$\begin{aligned} y \log x &= x - y \\ \Rightarrow y(1 + \log x) &= x \end{aligned}$$

i.e.  $y = \frac{x}{1+\log x}$

Differentiating both sides w.r.t.  $x$ , we get

$$\frac{dy}{dx} = \frac{(1+\log x) \cdot 1 - x \cdot \frac{1}{x}}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

**Example 14** If  $y = \tan x + \sec x$ , prove that  $\frac{d^2 y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$ .

**Solution** We have  $y = \tan x + \sec x$ . Differentiating w.r.t.  $x$ , we get

$$\begin{aligned} \frac{dy}{dx} &= \sec^2 x + \sec x \tan x \\ &= \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1+\sin x}{\cos^2 x} = \frac{1+\sin x}{(1+\sin x)(1-\sin x)} \end{aligned}$$

thus  $\frac{dy}{dx} = \frac{1}{1-\sin x}$ .

Now, differentiating again w.r.t.  $x$ , we get

$$\frac{d^2 y}{dx^2} = \frac{-(-\cos x)}{(1-\sin x)^2} = \frac{\cos x}{(1-\sin x)^2}$$

**Example 15** If  $f(x) = |\cos x|$ , find  $f' \frac{3\pi}{4}$ .

**Solution** When  $\frac{\pi}{2} < x < \pi$ ,  $\cos x < 0$  so that  $|\cos x| = -\cos x$ , i.e.,  $f(x) = -\cos x$   
 $f'(x) = \sin x$ .

Hence,  $f' \frac{3\pi}{4} = \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

**Example 16** If  $f(x) = |\cos x - \sin x|$ , find  $f' \frac{\pi}{6}$ .

**Solution** When  $0 < x < \frac{\pi}{4}$ ,  $\cos x > \sin x$ , so that  $\cos x - \sin x > 0$ , i.e.,

$$f(x) = \cos x - \sin x$$

$$f'(x) = -\sin x - \cos x$$

Hence  $f' \frac{\pi}{6} = -\sin \frac{\pi}{6} - \cos \frac{\pi}{6} = -\frac{1}{2}(1 + \sqrt{3})$ .

**Example 17** Verify Rolle's theorem for the function,  $f(x) = \sin 2x$  in  $0, \frac{\pi}{2}$ .

**Solution** Consider  $f(x) = \sin 2x$  in  $0, \frac{\pi}{2}$ . Note that:

- (i) The function  $f$  is continuous in  $0, \frac{\pi}{2}$ , as  $f$  is a sine function, which is always continuous.
- (ii)  $f'(x) = 2\cos 2x$ , exists in  $0, \frac{\pi}{2}$ , hence  $f$  is derivable in  $\left(0, \frac{\pi}{2}\right)$ .
- (iii)  $f(0) = \sin 0 = 0$  and  $f \frac{\pi}{2} = \sin \pi = 0 \Rightarrow f(0) = f \frac{\pi}{2}$ .

Conditions of Rolle's theorem are satisfied. Hence there exists at least one  $c \in 0, \frac{\pi}{2}$  such that  $f'(c) = 0$ . Thus

$$2 \cos 2c = 0 \quad \Rightarrow \quad 2c = \frac{\pi}{2} \quad \Rightarrow \quad c = \frac{\pi}{4}$$

**Example 18** Verify mean value theorem for the function  $f(x) = (x-3)(x-6)(x-9)$  in  $[3, 5]$ .

**Solution** (i) Function  $f$  is continuous in  $[3, 5]$  as product of polynomial functions is a polynomial, which is continuous.

(ii)  $f'(x) = 3x^2 - 36x + 99$  exists in  $(3, 5)$  and hence derivable in  $(3, 5)$ .

Thus conditions of mean value theorem are satisfied. Hence, there exists at least one  $c \in (3, 5)$  such that

$$\begin{aligned} f'(c) &= \frac{f(5) - f(3)}{5 - 3} \\ \Rightarrow 3c^2 - 36c + 99 &= \frac{8 - 0}{2} = 4 \\ \Rightarrow c &= 6 \pm \sqrt{\frac{13}{3}}. \end{aligned}$$

Hence  $c = 6 - \sqrt{\frac{13}{3}}$  (since other value is not permissible).

### Long Answer (L.A.)

**Example 19** If  $f(x) = \frac{\sqrt{2} \cos x - 1}{\cot x - 1}$ ,  $x \neq \frac{\pi}{4}$

find the value of  $f\left(\frac{\pi}{4}\right)$  so that  $f(x)$  becomes continuous at  $x = \frac{\pi}{4}$ .

**Solution** Given,  $f(x) = \frac{\sqrt{2} \cos x - 1}{\cot x - 1}$ ,  $x \neq \frac{\pi}{4}$

$$\begin{aligned} \text{Therefore, } \lim_{x \rightarrow \frac{\pi}{4}} f(x) &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sqrt{2} \cos x - 1}{\cot x - 1} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\sqrt{2} \cos x - 1) \sin x}{\cos x - \sin x} \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\sqrt{2} \cos x - 1)(\sqrt{2} \cos x + 1)}{(\sqrt{2} \cos x + 1)(\cos x - \sin x)} \cdot \frac{(\cos x + \sin x)}{(\cos x + \sin x)} \cdot \sin x \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow \frac{\pi}{4}} \frac{2 \cos^2 x - 1}{\cos^2 x - \sin^2 x} \cdot \frac{\cos x + \sin x}{\sqrt{2} \cos x + 1} \cdot (\sin x) \\
&= \lim_{x \rightarrow \frac{\pi}{4}} \frac{\cos 2x}{\cos 2x} \cdot \left( \frac{\cos x + \sin x}{\sqrt{2} \cos x + 1} \right) \cdot (\sin x) \\
&= \lim_{x \rightarrow \frac{\pi}{4}} \frac{(\cos x + \sin x) \sin x}{\sqrt{2} \cos x + 1} \\
&= \frac{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{\sqrt{2} \cdot \frac{1}{\sqrt{2}} + 1} = \frac{1}{2}
\end{aligned}$$

Thus,  $\lim_{x \rightarrow \frac{\pi}{4}} f(x) = \frac{1}{2}$

If we define  $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$ , then  $f(x)$  will become continuous at  $x = \frac{\pi}{4}$ . Hence for  $f$  to be continuous at  $x = \frac{\pi}{4}$ ,  $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$ .

**Example 20** Show that the function  $f$  given by  $f(x) = \begin{cases} \frac{e^x - 1}{e^x + 1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

is discontinuous at  $x = 0$ .

**Solution** The left hand limit of  $f$  at  $x = 0$  is given by

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{e^x - 1}{e^x + 1} = \frac{0 - 1}{0 + 1} = -1$$

Similarly, 
$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

$$= \lim_{x \rightarrow 0^+} \frac{1 - \frac{1}{e^{\frac{1}{x}}}}{1 + \frac{1}{e^{\frac{1}{x}}}} = \lim_{x \rightarrow 0^+} \frac{1 - e^{-\frac{1}{x}}}{1 + e^{-\frac{1}{x}}} = \frac{1 - 0}{1 + 0} = 1$$

Thus  $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$ , therefore,  $\lim_{x \rightarrow 0} f(x)$  does not exist. Hence  $f$  is discontinuous at  $x = 0$ .

**Example 21** Let  $f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2}, & \text{if } x < 0 \\ a, & \text{if } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4}, & \text{if } x > 0 \end{cases}$

For what value of  $a$ ,  $f$  is continuous at  $x = 0$ ?

**Solution** Here  $f(0) = a$  Left hand limit of  $f$  at 0 is

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \frac{1 - \cos 4x}{x^2} = \lim_{x \rightarrow 0^-} \frac{2 \sin^2 2x}{x^2} \\ &= \lim_{2x \rightarrow 0^-} 8 \frac{\sin^2 2x}{2x^2} = 8 (1)^2 = 8. \end{aligned}$$

and right hand limit of  $f$  at 0 is

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{16 + \sqrt{x}} - 4} \\ &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}(\sqrt{16 + \sqrt{x}} + 4)}{(\sqrt{16 + \sqrt{x}} + 4)(\sqrt{16 + \sqrt{x}} - 4)} \end{aligned}$$

$$= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{16+\sqrt{x}-16} = \lim_{x \rightarrow 0^+} (\sqrt{16+\sqrt{x}}+4) = 8$$

Thus,  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = 8$ . Hence  $f$  is continuous at  $x = 0$  only if  $a = 8$ .

**Example 22** Examine the differentiability of the function  $f$  defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } -3 \leq x < -2 \\ x+1, & \text{if } -2 \leq x < 0 \\ x+2, & \text{if } 0 \leq x \leq 1 \end{cases}$$

**Solution** The only doubtful points for differentiability of  $f(x)$  are  $x = -2$  and  $x = 0$ . Differentiability at  $x = -2$ .

$$\begin{aligned} \text{Now } Lf'(-2) &= \lim_{h \rightarrow 0^-} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{2(-2+h)+3 - (-2+1)}{h} = \lim_{h \rightarrow 0^-} \frac{2h}{h} = \lim_{h \rightarrow 0^-} 2 = 2. \end{aligned}$$

$$\begin{aligned} \text{and } Rf'(-2) &= \lim_{h \rightarrow 0^+} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{-2+h+1 - (-2+1)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h-1 - (-1)}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \end{aligned}$$

Thus  $Rf'(-2) \neq Lf'(-2)$ . Therefore  $f$  is not differentiable at  $x = -2$ . Similarly, for differentiability at  $x = 0$ , we have

$$\begin{aligned} L(f'(0)) &= \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{0+h+1 - (0+2)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{h-1}{h} = \lim_{h \rightarrow 0^-} 1 - \frac{1}{h} \end{aligned}$$

which does not exist. Hence  $f$  is not differentiable at  $x = 0$ .



**Example 23** Differentiate  $\tan^{-1} \frac{\sqrt{1-x^2}}{x}$  with respect to  $\cos^{-1} (2x\sqrt{1-x^2})$ , where

$$x \in \frac{1}{\sqrt{2}}, 1 .$$

**Solution** Let  $u = \tan^{-1} \frac{\sqrt{1-x^2}}{x}$  and  $v = \cos^{-1} (2x\sqrt{1-x^2})$ .

We want to find  $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

Now  $u = \tan^{-1} \frac{\sqrt{1-x^2}}{x}$ . Put  $x = \sin\theta$ ,  $\left(\frac{\pi}{4} < \theta < \frac{\pi}{2}\right)$ .

$$\begin{aligned} \text{Then } u &= \tan^{-1} \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} = \tan^{-1} (\cot\theta) \\ &= \tan^{-1} \left\{ \tan\left(\frac{\pi}{2}-\theta\right) \right\} = \frac{\pi}{2}-\theta = \frac{\pi}{2}-\sin^{-1}x \end{aligned}$$

$$\text{Hence } \frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}} .$$

$$\begin{aligned} \text{Now } v &= \cos^{-1} (2x\sqrt{1-x^2}) \\ &= \frac{\pi}{2} - \sin^{-1} (2x\sqrt{1-x^2}) \\ &= \frac{\pi}{2} - \sin^{-1} (2\sin\theta\sqrt{1-\sin^2\theta}) = \frac{\pi}{2} - \sin^{-1} (\sin 2\theta) \\ &= \frac{\pi}{2} - \sin^{-1} \{ \sin (\pi - 2\theta) \} \quad \left[ \text{since } \frac{\pi}{2} < 2\theta < \pi \right] \end{aligned}$$

$$= \frac{\pi}{2} - (\pi - 2\theta) = \frac{-\pi}{2} + 2\theta$$

$$\Rightarrow v = \frac{-\pi}{2} + 2\sin^{-1}x$$

$$\Rightarrow \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Hence

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = \frac{-1}{2}$$

### Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 24 to 35.

**Example 24** The function  $f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$

is continuous at  $x = 0$ , then the value of  $k$  is

- (A) 3 (B) 2  
(C) 1 (D) 1.5

**Solution** (B) is the Correct answer.

**Example 25** The function  $f(x) = [x]$ , where  $[x]$  denotes the greatest integer function, is continuous at

- (A) 4 (B) -2  
(C) 1 (D) 1.5

**Solution** (D) is the correct answer. The greatest integer function  $[x]$  is discontinuous at all integral values of  $x$ . Thus D is the correct answer.

**Example 26** The number of points at which the function  $f(x) = \frac{1}{x - [x]}$  is not continuous is

- (A) 1 (B) 2  
(C) 3 (D) none of these

**Solution** (D) is the correct answer. As  $x - [x] = 0$ , when  $x$  is an integer so  $f(x)$  is discontinuous for all  $x \in \mathbf{Z}$ .

**Example 27** The function given by  $f(x) = \tan x$  is discontinuous on the set

- (A)  $\{n\pi : n \in \mathbf{Z}\}$  (B)  $\{2n\pi : n \in \mathbf{Z}\}$   
 (C)  $(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}$  (D)  $\frac{n\pi}{2} : n \in \mathbf{Z}$

**Solution** C is the correct answer.

**Example 28** Let  $f(x) = |\cos x|$ . Then,

- (A)  $f$  is everywhere differentiable.  
 (B)  $f$  is everywhere continuous but not differentiable at  $x = n\pi, n \in \mathbf{Z}$ .  
 (C)  $f$  is everywhere continuous but not differentiable at  $x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}$ .  
 (D) none of these.

**Solution** C is the correct answer.

**Example 29** The function  $f(x) = |x| + |x-1|$  is

- (A) continuous at  $x=0$  as well as at  $x=1$ .  
 (B) continuous at  $x=1$  but not at  $x=0$ .  
 (C) discontinuous at  $x=0$  as well as at  $x=1$ .  
 (D) continuous at  $x=0$  but not at  $x=1$ .

**Solution** Correct answer is A.

**Example 30** The value of  $k$  which makes the function defined by

$$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}, \text{ continuous at } x = 0 \text{ is}$$

- (A) 8 (B) 1  
 (C) -1 (D) none of these

**Solution** (D) is the correct answer. Indeed  $\lim_{x \rightarrow 0} \sin \frac{1}{x}$  does not exist.

**Example 31** The set of points where the functions  $f$  given by  $f(x) = |x-3| \cos x$  is differentiable is

- (A)  $\mathbf{R}$  (B)  $\mathbf{R} - \{3\}$   
 (C)  $(0, \infty)$  (D) none of these

**Solution** B is the correct answer.

**Example 32** Differential coefficient of  $\sec(\tan^{-1}x)$  w.r.t.  $x$  is

- (A)  $\frac{x}{\sqrt{1+x^2}}$  (B)  $\frac{x}{1+x^2}$   
 (C)  $x\sqrt{1+x^2}$  (D)  $\frac{1}{\sqrt{1+x^2}}$

**Solution** (A) is the correct answer.

**Example 33** If  $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$  and  $v = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ , then  $\frac{du}{dv}$  is

- (A)  $\frac{1}{2}$  (B)  $x$  (C)  $\frac{1-x^2}{1+x^2}$  (D) 1

**Solution** (D) is the correct answer.

**Example 34** The value of  $c$  in Rolle's Theorem for the function  $f(x) = e^x \sin x$ ,  $x \in [0, \pi]$  is

- (A)  $\frac{\pi}{6}$  (B)  $\frac{\pi}{4}$  (C)  $\frac{\pi}{2}$  (D)  $\frac{3\pi}{4}$

**Solution** (D) is the correct answer.

**Example 35** The value of  $c$  in Mean value theorem for the function  $f(x) = x(x-2)$ ,  $x \in [1, 2]$  is

- (A)  $\frac{3}{2}$  (B)  $\frac{2}{3}$  (C)  $\frac{1}{2}$  (D)  $\frac{3}{2}$

**Solution** (A) is the correct answer.

**Example 36** Match the following

**COLUMN-I**

**COLUMN-II**

- (A) If a function  $f(x) = \begin{cases} \frac{\sin 3x}{x}, & \text{if } x \neq 0 \\ \frac{k}{2}, & \text{if } x = 0 \end{cases}$

(a)  $|x|$

is continuous at  $x = 0$ , then  $k$  is equal to

- (B) Every continuous function is differentiable (b) True  
 (C) An example of a function which is continuous everywhere but not differentiable at exactly one point (c) 6  
 (D) The identity function i.e.  $f(x) = x \forall x \in \mathbb{R}$  is a continuous function (d) False

**Solution**  $A \rightarrow c, B \rightarrow d, C \rightarrow a, D \rightarrow b$

Fill in the blanks in each of the Examples 37 to 41.

**Example 37** The number of points at which the function  $f(x) = \frac{1}{\log|x|}$  is discontinuous is \_\_\_\_\_.

**Solution** The given function is discontinuous at  $x = 0, \pm 1$  and hence the number of points of discontinuity is 3.

**Example 38** If  $f(x) = \begin{cases} ax+1 & \text{if } x \geq 1 \\ x+2 & \text{if } x < 1 \end{cases}$  is continuous, then  $a$  should be equal to \_\_\_\_\_.

**Solution**  $a = 2$

**Example 39** The derivative of  $\log_{10} x$  w.r.t.  $x$  is \_\_\_\_\_.

**Solution**  $(\log_{10} e) \frac{1}{x}$ .

**Example 40** If  $y = \sec^{-1} \left( \frac{\sqrt{x}+1}{\sqrt{x}-1} \right) + \sin^{-1} \left( \frac{\sqrt{x}-1}{\sqrt{x}+1} \right)$ , then  $\frac{dy}{dx}$  is equal to \_\_\_\_\_.

**Solution** 0.

**Example 41** The derivative of  $\sin x$  w.r.t.  $\cos x$  is \_\_\_\_\_.

**Solution**  $-\cot x$

State whether the statements are True or False in each of the Exercises 42 to 46.

**Example 42** For continuity, at  $x = a$ , each of  $\lim_{x \rightarrow a^+} f(x)$  and  $\lim_{x \rightarrow a^-} f(x)$  is equal to  $f(a)$ .

**Solution** True.

**Example 43**  $y = |x - 1|$  is a continuous function.

**Solution** True.

**Example 44** A continuous function can have some points where limit does not exist.

**Solution** False.

**Example 45**  $|\sin x|$  is a differentiable function for every value of  $x$ .

**Solution** False.

**Example 46**  $\cos |x|$  is differentiable everywhere.

**Solution** True.

### 5.3 EXERCISE

#### Short Answer (S.A.)

1. Examine the continuity of the function

$$f(x) = x^3 + 2x^2 - 1 \text{ at } x = 1$$

Find which of the functions in Exercises 2 to 10 is continuous or discontinuous at the indicated points:

2.  $f(x) = \begin{cases} 3x+5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$

at  $x=2$

3.  $f(x) = \begin{cases} \frac{1 - \cos 2x}{x^2}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases}$

at  $x=0$

4.  $f(x) = \begin{cases} \frac{2x^2 - 3x - 2}{x - 2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$

at  $x=2$

5.  $f(x) = \begin{cases} \frac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$

at  $x=4$

6.  $f(x) = \begin{cases} |x| \cos \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

at  $x=0$

7.  $f(x) = \begin{cases} |x-a| \sin \frac{1}{x-a}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$

at  $x=a$

8.  $f(x) = \begin{cases} \frac{1}{e^x}, & \text{if } x \neq 0 \\ 1 + e^{\frac{1}{x}}, & \text{if } x = 0 \end{cases}$

at  $x=0$

9.  $f(x) = \begin{cases} \frac{x^2}{2}, & \text{if } 0 \leq x \leq 1 \\ 2x^2 - 3x + \frac{3}{2}, & \text{if } 1 < x \leq 2 \end{cases}$

at  $x=1$

10.  $f(x) = |x| + |x-1|$  at  $x=1$

Find the value of  $k$  in each of the Exercises 11 to 14 so that the function  $f$  is continuous at the indicated point:

$$11. f(x) = \begin{cases} 3x-8, & \text{if } x \leq 5 \\ 2k, & \text{if } x > 5 \end{cases} \quad \text{at } x=5 \qquad 12. f(x) = \begin{cases} \frac{2^{x+2}-16}{4^x-16}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases} \quad \text{at } x=2$$

$$13. f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \leq x < 0 \\ \frac{2x+1}{x-1}, & \text{if } 0 \leq x \leq 1 \end{cases} \quad \text{at } x = 0$$

$$14. f(x) = \begin{cases} \frac{1-\cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \frac{1}{2}, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0$$

15. Prove that the function  $f$  defined by

$$f(x) = \begin{cases} \frac{x}{|x|+2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

remains discontinuous at  $x = 0$ , regardless the choice of  $k$ .

16. Find the values of  $a$  and  $b$  such that the function  $f$  defined by

$$f(x) = \begin{cases} \frac{x-4}{|x-4|} + a, & \text{if } x < 4 \\ a+b, & \text{if } x = 4 \\ \frac{x-4}{|x-4|} + b, & \text{if } x > 4 \end{cases}$$

is a continuous function at  $x = 4$ .

17. Given the function  $f(x) = \frac{1}{x+2}$ . Find the points of discontinuity of the composite function  $y = f(f(x))$ .

18. Find all points of discontinuity of the function  $f(t) = \frac{1}{t^2 + t - 2}$ , where  $t = \frac{1}{x-1}$ .

19. Show that the function  $f(x) = |\sin x + \cos x|$  is continuous at  $x = \pi$ .

Examine the differentiability of  $f$ , where  $f$  is defined by

$$20. f(x) = \begin{cases} x[x], & \text{if } 0 \leq x < 2 \\ (x-1)x, & \text{if } 2 \leq x < 3 \end{cases}$$

at  $x = 2$ .

$$21. f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

at  $x = 0$ .

$$22. f(x) = \begin{cases} 1+x & \text{if } x \leq 2 \\ 5-x & \text{if } x > 2 \end{cases}$$

at  $x = 2$ .

23. Show that  $f(x) = |x-5|$  is continuous but not differentiable at  $x = 5$ .

24. A function  $f: \mathbf{R} \rightarrow \mathbf{R}$  satisfies the equation  $f(x+y) = f(x)f(y)$  for all  $x, y \in \mathbf{R}$ ,  $f(x) \neq 0$ . Suppose that the function is differentiable at  $x = 0$  and  $f'(0) = 2$ . Prove that  $f'(x) = 2f(x)$ .

Differentiate each of the following w.r.t.  $x$  (Exercises 25 to 43):

25.  $2^{\cos^2 x}$

26.  $\frac{8^x}{x^8}$

27.  $\log(x + \sqrt{x^2 + a})$

28.  $\log[\log(\log x^5)]$

29.  $\sin \sqrt{x} + \cos^2 \sqrt{x}$

30.  $\sin^n(ax^2 + bx + c)$

31.  $\cos(\tan \sqrt{x+1})$

32.  $\sin x^2 + \sin^2 x + \sin^2(x^2)$

33.  $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$

34.  $(\sin x)^{\cos x}$

35.  $\sin^m x \cdot \cos^n x$

36.  $(x+1)^2 (x+2)^3 (x+3)^4$



37.  $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), \frac{-\pi}{4} < x < \frac{\pi}{4}$
38.  $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$
39.  $\tan^{-1}(\sec x + \tan x), -\frac{\pi}{2} < x < \frac{\pi}{2}$
40.  $\tan^{-1}\left(\frac{a \cos x - b \sin x}{b \cos x + a \sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$  and  $\frac{a}{b} \tan x > -1$
41.  $\sec^{-1}\left(\frac{1}{4x^3 - 3x}\right), 0 < x < \frac{1}{\sqrt{2}}$
42.  $\tan^{-1} \frac{3a^2x - x^3}{a^3 - 3ax^2}, \frac{-1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$
43.  $\tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right), -1 < x < 1, x \neq 0$

Find  $\frac{dy}{dx}$  of each of the functions expressed in parametric form in Exercises from 44 to 48.

44.  $x = t + \frac{1}{t}, y = t - \frac{1}{t}$
45.  $x = e^\theta \left(\theta + \frac{1}{\theta}\right), y = e^{-\theta} \left(\theta - \frac{1}{\theta}\right)$
46.  $x = 3\cos\theta - 2\cos^3\theta, y = 3\sin\theta - 2\sin^3\theta$
47.  $\sin x = \frac{2t}{1+t^2}, \tan y = \frac{2t}{1-t^2}$
48.  $x = \frac{1+\log t}{t^2}, y = \frac{3+2\log t}{t}$
49. If  $x = e^{\cos 2t}$  and  $y = e^{\sin 2t}$ , prove that  $\frac{dy}{dx} = \frac{-y \log x}{x \log y}$ .
50. If  $x = a \sin 2t (1 + \cos 2t)$  and  $y = b \cos 2t (1 - \cos 2t)$ , show that  $\left(\frac{dy}{dx}\right)_{at=\frac{\pi}{4}} = \frac{b}{a}$ .
51. If  $x = 3\sin t - \sin 3t, y = 3\cos t - \cos 3t$ , find  $\frac{dy}{dx}$  at  $t = \frac{\pi}{3}$ .

52. Differentiate  $\frac{x}{\sin x}$  w.r.t.  $\sin x$ .

53. Differentiate  $\tan^{-1} \left( \frac{\sqrt{1+x^2}-1}{x} \right)$  w.r.t.  $\tan^{-1} x$  when  $x \neq 0$ .

Find  $\frac{dy}{dx}$  when  $x$  and  $y$  are connected by the relation given in each of the Exercises 54 to 57.

54.  $\sin(xy) + \frac{x}{y} = x^2 - y$

55.  $\sec(x+y) = xy$

56.  $\tan^{-1}(x^2 + y^2) = a$

57.  $(x^2 + y^2)^2 = xy$

58. If  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ , then show that  $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ .

59. If  $x = e^{\frac{x}{y}}$ , prove that  $\frac{dy}{dx} = \frac{x-y}{x \log x}$ .

60. If  $y^x = e^{y-x}$ , prove that  $\frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y}$ .

61. If  $y = (\cos x)^{(\cos x)^{(\cos x) \dots \infty}}$ , show that  $\frac{dy}{dx} = \frac{y^2 \tan x}{y \log \cos x - 1}$ .

62. If  $x \sin(a+y) + \sin a \cos(a+y) = 0$ , prove that  $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$ .

63. If  $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ , prove that  $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ .

64. If  $y = \tan^{-1} x$ , find  $\frac{d^2 y}{dx^2}$  in terms of  $y$  alone.

Verify the Rolle's theorem for each of the functions in Exercises 65 to 69.

65.  $f(x) = x(x-1)^2$  in  $[0, 1]$ .

66.  $f(x) = \sin^4 x + \cos^4 x$  in  $\left[0, \frac{\pi}{2}\right]$ .

67.  $f(x) = \log(x^2 + 2) - \log 3$  in  $[-1, 1]$ .

68.  $f(x) = x(x+3)e^{-x/2}$  in  $[-3, 0]$ .

69.  $f(x) = \sqrt{4-x^2}$  in  $[-2, 2]$ .

70. Discuss the applicability of Rolle's theorem on the function given by

$$f(x) = \begin{cases} x^2 + 1, & \text{if } 0 \leq x \leq 1 \\ 3 - x, & \text{if } 1 \leq x \leq 2 \end{cases}$$

71. Find the points on the curve  $y = (\cos x - 1)$  in  $[0, 2\pi]$ , where the tangent is parallel to  $x$ -axis.

72. Using Rolle's theorem, find the point on the curve  $y = x(x-4)$ ,  $x \in [0, 4]$ , where the tangent is parallel to  $x$ -axis.

Verify mean value theorem for each of the functions given Exercises 73 to 76.

73.  $f(x) = \frac{1}{4x-1}$  in  $[1, 4]$ .

74.  $f(x) = x^3 - 2x^2 - x + 3$  in  $[0, 1]$ .

75.  $f(x) = \sin x - \sin 2x$  in  $[0, \pi]$ .

76.  $f(x) = \sqrt{25-x^2}$  in  $[1, 5]$ .

77. Find a point on the curve  $y = (x-3)^2$ , where the tangent is parallel to the chord joining the points  $(3, 0)$  and  $(4, 1)$ .

78. Using mean value theorem, prove that there is a point on the curve  $y = 2x^2 - 5x + 3$  between the points A(1, 0) and B(2, 1), where tangent is parallel to the chord AB. Also, find that point.

### Long Answer (L.A.)

79. Find the values of  $p$  and  $q$  so that

$$f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \leq 1 \\ qx + 2 & , \text{if } x > 1 \end{cases}$$

is differentiable at  $x = 1$ .

**80.** If  $x^m \cdot y^n = (x + y)^{m+n}$ , prove that

(i)  $\frac{dy}{dx} = \frac{y}{x}$  and (ii)  $\frac{d^2y}{dx^2} = 0$ .

**81.** If  $x = \sin t$  and  $y = \sin pt$ , prove that  $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + p^2 y = 0$ .

**82.** Find  $\frac{dy}{dx}$ , if  $y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}$ .

### Objective Type Questions

Choose the correct answers from the given four options in each of the Exercises 83 to 96.

**83.** If  $f(x) = 2x$  and  $g(x) = \frac{x^2}{2} + 1$ , then which of the following can be a discontinuous function

(A)  $f(x) + g(x)$

(B)  $f(x) - g(x)$

(C)  $f(x) \cdot g(x)$

(D)  $\frac{g(x)}{f(x)}$

**84.** The function  $f(x) = \frac{4-x^2}{4x-x^3}$  is

(A) discontinuous at only one point

(B) discontinuous at exactly two points

(C) discontinuous at exactly three points

(D) none of these

**85.** The set of points where the function  $f$  given by  $f(x) = |2x-1| \sin x$  is differentiable is

(A)  $\mathbf{R}$

(B)  $\mathbf{R} - \left\{ \frac{1}{2} \right\}$

- (C)  $(0, \infty)$  (D) none of these
- 86.** The function  $f(x) = \cot x$  is discontinuous on the set
- (A)  $\{x = n\pi : n \in \mathbf{Z}\}$  (B)  $\{x = 2n\pi : n \in \mathbf{Z}\}$
- (C)  $\left\{x = (2n+1)\frac{\pi}{2} ; n \in \mathbf{Z}\right\}$  (iv)  $\left\{x = \frac{n\pi}{2} ; n \in \mathbf{Z}\right\}$
- 87.** The function  $f(x) = e^{|x|}$  is
- (A) continuous everywhere but not differentiable at  $x = 0$   
 (B) continuous and differentiable everywhere  
 (C) not continuous at  $x = 0$   
 (D) none of these.
- 88.** If  $f(x) = x^2 \sin \frac{1}{x}$ , where  $x \neq 0$ , then the value of the function  $f$  at  $x = 0$ , so that the function is continuous at  $x = 0$ , is
- (A) 0 (B) -1  
 (C) 1 (D) none of these
- 89.** If  $f(x) = \begin{cases} mx+1 & , \text{if } x \leq \frac{\pi}{2} \\ \sin x + n, & \text{if } x > \frac{\pi}{2} \end{cases}$ , is continuous at  $x = \frac{\pi}{2}$ , then
- (A)  $m = 1, n = 0$  (B)  $m = \frac{n\pi}{2} + 1$
- (C)  $n = \frac{m\pi}{2}$  (D)  $m = n = \frac{\pi}{2}$
- 90.** Let  $f(x) = |\sin x|$ . Then
- (A)  $f$  is everywhere differentiable  
 (B)  $f$  is everywhere continuous but not differentiable at  $x = n\pi, n \in \mathbf{Z}$ .  
 (C)  $f$  is everywhere continuous but not differentiable at  $x = (2n + 1) \frac{\pi}{2}$ ,  
 $n \in \mathbf{Z}$ .  
 (D) none of these
- 91.** If  $y = \log \left( \frac{1-x^2}{1+x^2} \right)$ , then  $\frac{dy}{dx}$  is equal to

- (A)  $\frac{4x^3}{1-x^4}$  (B)  $\frac{-4x}{1-x^4}$   
 (C)  $\frac{1}{4-x^4}$  (D)  $\frac{-4x^3}{1-x^4}$

92. If  $y = \sqrt{\sin x + y}$ , then  $\frac{dy}{dx}$  is equal to

- (A)  $\frac{\cos x}{2y-1}$  (B)  $\frac{\cos x}{1-2y}$   
 (C)  $\frac{\sin x}{1-2y}$  (D)  $\frac{\sin x}{2y-1}$

93. The derivative of  $\cos^{-1}(2x^2 - 1)$  w.r.t.  $\cos^{-1}x$  is

- (A) 2 (B)  $\frac{-1}{2\sqrt{1-x^2}}$   
 (C)  $\frac{2}{x}$  (D)  $1 - x^2$

94. If  $x = t^2$ ,  $y = t^3$ , then  $\frac{d^2y}{dx^2}$  is

- (A)  $\frac{3}{2}$  (B)  $\frac{3}{4t}$   
 (C)  $\frac{3}{2t}$  (D)  $\frac{3}{4}$

95. The value of  $c$  in Rolle's theorem for the function  $f(x) = x^3 - 3x$  in the interval  $[0, \sqrt{3}]$  is

- (A) 1 (B) -1

(C)  $\frac{3}{2}$

(D)  $\frac{1}{3}$

96. For the function  $f(x) = x + \frac{1}{x}$ ,  $x \in [1, 3]$ , the value of  $c$  for mean value theorem is

(A) 1

(B)  $\sqrt{3}$

(C) 2

(D) none of these

Fill in the blanks in each of the Exercises 97 to 101:

97. An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is \_\_\_\_\_.

98. Derivative of  $x^2$  w.r.t.  $x^3$  is \_\_\_\_\_.

99. If  $f(x) = |\cos x|$ , then  $f' \frac{\pi}{4} =$  \_\_\_\_\_.

100. If  $f(x) = |\cos x - \sin x|$ , then  $f' \frac{\pi}{3} =$  \_\_\_\_\_.

101. For the curve  $\sqrt{x} + \sqrt{y} = 1$ ,  $\frac{dy}{dx}$  at  $\left(\frac{1}{4}, \frac{1}{4}\right)$  is \_\_\_\_\_.

State **True** or **False** for the statements in each of the Exercises 102 to 106.

102. Rolle's theorem is applicable for the function  $f(x) = |x - 1|$  in  $[0, 2]$ .

103. If  $f$  is continuous on its domain  $D$ , then  $|f|$  is also continuous on  $D$ .

104. The composition of two continuous function is a continuous function.

105. Trigonometric and inverse - trigonometric functions are differentiable in their respective domain.

106. If  $f, g$  is continuous at  $x = a$ , then  $f$  and  $g$  are separately continuous at  $x = a$ .

